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1993 J. Phys. A: Math. Gen. 26 3837

(http://iopscience.iop.org/0305-4470/26/15/031)

J. Phys. A: Math. Gen. 26 (1993) 3837–3844. Printed in the UK

Variational approach to the Bogomolny separation

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Received 3 March 1993

Abstract. An alternative to the Bogomolny separation for the three-dimensional Heisenberg model is studied. An example of an alternative approach to one- and two-dimensional spin field models is given. Energetic stability of ferromagnet configurations with non-trivial Hopf index within isotropic Heisenberg model is considered.

1. Introduction

The possibility of the existence of topologically stable configurations of the Heisenberg ferromagnet has been discussed over past two decades [1–5].

In the Heisenberg model of a ferromagnet we treat magnetization as a continuous unit vector field in a three-dimensional space $\mathcal{R}^3$, $|m(x, y, z)| = 1$. The Heisenberg Hamiltonian of field m in an isotropic ferromagnet is defined as

$$H = \int_{\mathcal{R}^3} (\nabla m)^2 \, d^3x. \tag{1}$$

In order to discuss the topological properties of field m, we need to redefine it on sphere S^3 in the four-dimensional space $\mathcal{R}^4$. This is done by composing a stereographic projection from S^3 to $\mathcal{R}^3$ with the original map m resulting in a unit vector field defined on S^3. As there is one-to-one correspondence between all unit vectors in $\mathcal{R}^3$ and points on two-dimensional sphere S^2, the configuration of the whole ferromagnet can be identified with map S defined as follows

$$S : S^3 \mapsto S^2. \tag{2}$$

One of the fundamental properties of map (2) is its *Hopf index* defined as an integral (see (29)) taking on integer values only. The index is insensitive to continuous changes of map S and thus it is called a *topological invariant* and can be used to classify all mappings (2) [6, 7].

Consequently, no continuous transition from one Hopf class to another one is possible and it is this property which protects a configuration from 'flattening' itself (i.e. evolving to configuration with constant magnetization vector).

This paper is devoted to a study of the existence of configurations with minimum energy within a given Hopf class. The plan of the presentation is as follows. In the next section two examples of the conditional minimization of the energy functional used in section 4 are studied. The Hopf index is defined in section 3 and the main result proving the non-existence of a minimum energy configuration for a given Hopf class is established.

2. Alternatives to the Bogomolny separation for the classical spin field

In the case of a topologically stable configuration we deal with a problem of the minimization of a functional (Hamiltonian) within the space of constant topological charge. If we take the typical approach which is to search for the functional's minimum, we will find that the topological part of the functional does not contribute to the resulting equations. This is caused by the vanishing of the functional derivative of the topological part of the Hamiltonian.

In order to solve the problem we choose, as our new variables, functions whose variations may lead to global changes in the field configuration. Such a selection, in general, leads to more equations than is necessary. The number of these equations may then be reduced by means of selecting the appropriate value for the spectral parameter (Lagrange's multiplier). The choice of the new variables is neither unique nor simple. In general, selection should be based on derivatives of the field variables in such a way that the variational derivative of the topological charge with respect to the new set of variables does not vanish and it should be possible to write the Hamiltonian in terms of the new variables.

As a simple example we will discuss the case of a two-dimensional spin field on one-dimensional compact space (circular chain of spins). The topological charge of the field configuration is represented by number of turns of spin vector along the chain. A classical Hamiltonian for such a field reads as follows

$$H = \int_{-\infty}^{\infty} \left(\frac{\partial S}{\partial l}\right)^2 \mathrm{d}l \tag{3}$$

and may serve as a very good example for displaying the principles of the proposed method.

We wish to find the minimum of Hamiltonian (3) under the condition of constant value of integral:

$$\int_{-\infty}^{\infty} \frac{\partial \tan^{-1}(S^y/S^x)}{\partial l}\, \mathrm{d}l \tag{4}$$

representing the total number of turns vector S has done.

Employing the standard Lagrange multiplier method we introduce the Lagrange functional as

$$\mathcal{H} = \int_{-\infty}^{\infty} \left(\frac{\partial S}{\partial l}\right)^2 \mathrm{d}l - \lambda \int_{-\infty}^{\infty} \frac{\partial \tan^{-1}(S^y/S^x)}{\partial l}\, \mathrm{d}l. \tag{5}$$

Introducing the stereographical coordinate u:

$$S_x = \frac{2u}{1+u^2} \qquad S_y = \frac{1-u^2}{1+u^2} \tag{6}$$

we express the Hamiltonian in terms of coordinate u in the following form:

$$\mathcal{H} = \int_{-\infty}^{\infty} \frac{4(u')^2}{(1+u^2)^2}\, \mathrm{d}l - \int_{-\infty}^{\infty} \frac{2\lambda u'}{1+u^2}\, \mathrm{d}l. \tag{7}$$

Denoting $\alpha = 2u'/(1+u^2)$ we end up with the simple formula:

$$\mathcal{H} = \int_{-\infty}^{\infty} \alpha^2\, \mathrm{d}l - \lambda \int_{-\infty}^{\infty} \alpha\, \mathrm{d}l. \tag{8}$$

Taking variation of $\mathcal{H}$ with respect to variable α we obtain the differential equation:

$$2\alpha - \lambda = \frac{4u'}{1+u^2} - \lambda = 0 \tag{9}$$

with the corresponding solution as follows

$$u = \tan(\tfrac{1}{4}\lambda l + c). \tag{10}$$

This leads to the following formula for $\boldsymbol{S}$:

$$\boldsymbol{S} = \{\sin(\tfrac{1}{4}\lambda l + c), \cos(\tfrac{1}{4}\lambda l + c)\}. \tag{11}$$

The less trivial problem of a two-dimensional σ-model has been solved in [8, 9]. The generalized (with topological charge) σ-model Hamiltonian reads as follows

$$\mathcal{H} = H(S) - \lambda I(S) \tag{12}$$

$$H(S) = \int S^{\alpha}_{,i} S^{\alpha}_{,i} \, \mathrm{d}^2 x \tag{13}$$

$$I(S) = \int \epsilon^{\alpha\beta\gamma} \epsilon_{ab} S^{\alpha} S^{\beta}_{,a} S^{\gamma}_{,b} \, \mathrm{d}^2 x. \tag{14}$$

In stereographic variables:

$$w = u + \mathrm{i}v = \cot(\theta/2)\mathrm{e}^{\mathrm{i}\phi}. \tag{15}$$

The Hamiltonian density takes the form

$$h = \frac{\nabla w \nabla w^* - \lambda \mathrm{i}(w_{,1} w^*_{,2} - w_{,2} w^*_{,1})}{(1 + ww^*)^2}. \tag{16}$$

Selecting new variables as follows

$$\boldsymbol{v} = \frac{[(\mathrm{Re}(w))_{,1}, (\mathrm{Re}(w))_{,2}, (\mathrm{Im}(w))_{,1}, (\mathrm{Im}(w))_{,2}]}{1 + ww^*} \tag{17}$$

we end up with the Hamiltonian density in the following form:

$$h = |\boldsymbol{v}|^2 - \lambda(v^3 v^2 - v^1 v^4). \tag{18}$$

The minimization condition $\delta\mathcal{H}/\delta\boldsymbol{v} = 0$ leads us to the set of linear equations:

$$D\boldsymbol{v} = \lambda \boldsymbol{v} \tag{19}$$

with the corresponding characteristic equation:

$$\det(D - \lambda E) = 0 \tag{20}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

This leads (taking into account the definitions of $\boldsymbol{v}$ (17) and w (15)) to Riemann–Cauchy conditions and thus to solutions in the form of analytic functions of $z = x + \mathrm{i}y$ for an eigenvalue of equation (19) $\lambda_1 = 0$. The second eigenvalue $\lambda_2 = 2$ corresponds to analytic functions of $z = x - \mathrm{i}y$. Thus eigenvalues λ_1, λ_2 correspond respectively to instanton and anti-instanton solutions.

3. Hopf index—a topological invariant for field $m : \mathcal{R}^3 \mapsto S^2$

Let us consider the three-dimensional vector field $m : \mathcal{R}^3 \mapsto S^2$; $|m| = 1$. After compactification of $\mathcal{R}^3$ we end up with field $S : S^3 \mapsto S^2$ for which we can define a Hopf index [7].

First we have to introduce:

(i) the form σ_2 on S^2 such that

$$\sigma_2 : S^2 \mapsto \mathcal{R}^3 \tag{22}$$

and

$$\int_{S^2} \sigma_2 = 1 \tag{23}$$

(ii) the form ω:

$$\omega = S^* \sigma_2 \tag{24}$$

(iii) with corresponding form α such that

$$\omega = \mathrm{d}\alpha. \tag{25}$$

By writing (24) explicitly, we obtain

$$\omega = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \epsilon^{klm} \epsilon_{mpq} S^\alpha S^\beta_{,k} S^\gamma_{,l} \, \mathrm{d}x^p \, \mathrm{d}x^q \tag{26}$$

and we can calculate α, using the local reverse of the Poincaré lemma [7], as

$$\alpha = \epsilon_{klm} \left[\int_0^1 \mathcal{A}^k(tx) t \, \mathrm{d}t \right] x^l \, \mathrm{d}x^m \tag{27}$$

where

$$\mathcal{A}^m = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \epsilon^{klm} S^\alpha S^\beta_{,k} S^\gamma_{,l}. \tag{28}$$

Consequently, the Hopf index in terms of α and ω is

$$q = \int_{S^3} \alpha \wedge \omega. \tag{29}$$

This formal definition can be rewritten in a more practical form using vector potential A_μ and current J_μ [6]:

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} S^a \partial_\nu S^b \partial_\lambda S^c \tag{30}$$

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \tag{31}$$

leading to

$$q = -\int_{\mathcal{R}^3} \mathrm{d}^3x \, A_\mu J^\mu. \tag{32}$$

The relationship between J_μ, A_μ and coefficients of the forms α, ω and between J_μ and A_μ read as follows

$$\omega = \tfrac{1}{2}\epsilon_{\mu\nu\lambda} J^\mu \, \mathrm{d}x^\nu \, \mathrm{d}x^\lambda \tag{33}$$

$$\alpha = A_\mu \, \mathrm{d}x^\mu \tag{34}$$

$$A^\lambda = \epsilon_{\lambda\mu\nu} x^\nu \int_0^1 J^\mu(t\boldsymbol{x}) t \, \mathrm{d}t. \tag{35}$$

To simplify the resulting formula for q we project sphere S^2 by stereographical projection to two-dimensional Euclidean space $\mathcal{R}^2$ with coordinates (u, v):

$$S_x = \frac{2u}{1+u^2+v^2} \qquad S_y = \frac{2v}{1+u^2+v^2} \qquad S_z = \frac{1-u^2-v^2}{1+u^2+v^2} \tag{36}$$

leading us to an expression for J^μ:

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} \frac{(u_{'\nu} v_{'\lambda} - u_{'\lambda} v_{'\nu})}{(1+u^2+v^2)^2}. \tag{37}$$

Denoting

$$\alpha_\mu = \frac{u_{'\mu}}{(1+u^2+v^2)} \tag{38}$$

$$\beta_\mu = \frac{v_{'\mu}}{(1+u^2+v^2)} \tag{39}$$

we obtain an expression for Hopf index q:

$$q = -\frac{1}{\pi^2}\epsilon_{\mu\rho\sigma}\epsilon^{\mu\nu\lambda}\epsilon^{\rho\eta\xi} \int_{\mathcal{R}^3} \mathrm{d}^3x \, x^\sigma \alpha_\nu(\boldsymbol{x})\beta_\lambda(\boldsymbol{x}) \int_0^1 \alpha_\eta(t\boldsymbol{x})\beta_\xi(t\boldsymbol{x}) t \, \mathrm{d}t. \tag{40}$$

So for a given spin field $\boldsymbol{m}$ we can calculate the corresponding Hopf index q.

An example of a non-trivial configuration is provided by the function $w = u/v$ where $u = \alpha + \mathrm{i}\beta$ and $v = \gamma + \mathrm{i}\delta$ [6]. Coordinates $\alpha, \beta, \gamma, \delta$ are defined on the S^3 sphere. This surface is an image of a three-dimensional Euclidean space (x, y, z) in stereographic projection:

$$\begin{aligned} \alpha &= \frac{2x}{1+x^2+y^2+z^2} & \beta &= \frac{2y}{1+x^2+y^2+z^2} \\ \gamma &= \frac{2z}{1+x^2+y^2+z^2} & \delta &= \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}. \end{aligned} \tag{41}$$

Coordinates of vector S are represented by complex number w using formula (36) with $u = \mathrm{Re}(w)$ and $v = \mathrm{Im}(w)$. The Hopf index for this configuration can be easily calculated. The formulae for the current and vector potentials are

$$J(x, y, z) = \left\{ \begin{array}{c} \dfrac{8(y+xz)}{\pi(1+x^2+y^2+z^2)^3} \\ \dfrac{-8(x-yz)}{\pi(1+x^2+y^2+z^2)^3} \\ \dfrac{-4(-1+x^2+y^2-z^2)}{\pi(1+x^2+y^2+z^2)^3} \end{array} \right\} \tag{42}$$

$$A_\mu = \mathrm{i}z^\dagger \partial_\mu z / 2\pi \tag{43}$$

$$z = \begin{pmatrix} \alpha + \mathrm{i}\beta \\ \delta + \mathrm{i}\gamma \end{pmatrix} \tag{44}$$

and the dot product in (32) reads as

$$A_\mu J^\mu = \frac{-4}{\pi^2(1+x^2+y^2+z^2)^3}. \tag{45}$$

This leads us to the following formula in spherical coordinates for the Hopf index:

$$q = \int_0^\infty \mathrm{d}r\, \frac{16r^2}{\pi(1+r^2)^3} = 1. \tag{46}$$

4. Equation for topologically stable configurations of field S

The Hamiltonian (1) can now be rewritten in terms of α_μ and β_ν:

$$H = \int_{\mathcal{R}^3} \mathrm{d}^3x\, (\alpha_\mu\alpha^\mu + \beta_\mu\beta^\mu). \tag{47}$$

In order to obtain an equation for minimal energy with constant Hopf index, we calculate the variational derivative of $\mathcal{H}$:

$$\mathcal{H} = H + \lambda q \tag{48}$$

with respect to functions α_μ and β_ν, using formula (40) for the topological part of $\mathcal{H}$. This leads to the following set of equations:

$$\frac{\delta\mathcal{H}}{\delta\alpha_\tau(y)} = 2\alpha^\tau(y) - \frac{\lambda}{\pi^2} y^\sigma \epsilon_{\mu\rho\sigma}\epsilon^{\rho\eta\xi}\epsilon^{\mu\tau\lambda}\beta_\lambda(y) \int_0^\infty \alpha_\eta(ty)\beta_\xi(ty) t\, \mathrm{sgn}(1-t)\, \mathrm{d}t = 0 \tag{49}$$

with an analogous equation for the derivative with respect to $\beta_\tau(y)$:

$$\frac{\delta\mathcal{H}}{\delta\beta_\tau(y)} = 2\beta^\tau(y) + \frac{\lambda}{\pi^2} y^\sigma \epsilon_{\mu\rho\sigma}\epsilon^{\rho\eta\xi}\epsilon^{\mu\tau\lambda}\alpha_\lambda(y) \int_0^\infty \alpha_\eta(ty)\beta_\xi(ty) t\, \mathrm{sgn}(1-t)\, \mathrm{d}t = 0. \tag{50}$$

These formulae are equivalent to the following simplified ones (these equations play a similar role to that of equation (19) in the σ-model example):

$$\boldsymbol{\alpha} = -\frac{\lambda}{2\pi}(\boldsymbol{I}\times\boldsymbol{x})\times\boldsymbol{\beta} \qquad \boldsymbol{\beta} = \frac{\lambda}{2\pi}(\boldsymbol{I}\times\boldsymbol{x})\times\boldsymbol{\alpha} \tag{51}$$

where

$$\boldsymbol{I} = \int_0^\infty [\boldsymbol{\alpha}(t\boldsymbol{x})\times\boldsymbol{\beta}(t\boldsymbol{x})] t\, \mathrm{sgn}(1-t)\, \mathrm{d}t. \tag{52}$$

Let us now analyse the overdetermined system of six equations (51) for functions u, v. We will now show, using the *reductio ad absurdum* method, that this system has no solutions. We shall try to reduce the number of equations by selecting a special value for the spectral parameter λ. Inserting the second equation into the first, after simple vector calculations, we obtain

$$\boldsymbol{\alpha}\left(1 - \left(\frac{\lambda}{2\pi}(\boldsymbol{I}\times\boldsymbol{x})\right)^2\right) = \left(\frac{\lambda}{2\pi}\right)^2 ((\boldsymbol{I}\times\boldsymbol{x})\cdot\boldsymbol{\alpha})(\boldsymbol{I}\times\boldsymbol{x}). \tag{53}$$

As we can see from the first line in (51) α must be perpendicular to $I \times x$ and the right-hand side of equation (53) is zero. Consequently equation (53) can be satisfied only if the coefficient on the left-hand side vanishes:

$$\left(\frac{\lambda}{2\pi}(I \times x)\right)^2 = 1 \tag{54}$$

or function α is equal to zero:

$$\alpha = 0. \tag{55}$$

Solution (55) leads to the trivial case of the constant field with Hopf index zero. Equation (54) is equivalent to the characteristic equation (20) from the two-dimensional example, except that it is a nonlinear equation for parameter λ and two functions α, β from (52). We can use this equation without actually solving it. Equation (54) means that vector $(\lambda/2\pi)(I \times x) = e$ is a unit vector, so we can use this information to eliminate the second equation from (51). Now we have set of four equations (three from (51) and the fourth one is (54)) for two functions u, v (38).

The left-hand side of equation (54) must be constant. We now introduce the notation:

$$f(x) = x \times (\alpha(x) \times \beta(x)) \tag{56}$$

$$g(x) = \left[\int_0^1 f(tx)\,\mathrm{d}t - \int_1^\infty f(tx)\,\mathrm{d}t\right]. \tag{57}$$

A necessary condition for (54) to be satisfied is $g(sx)^2 = g(x)^2$. Consequently

$$g(sx)^2 = \left[\int_0^1 f(stx)\,\mathrm{d}t - \int_1^\infty f(stx)\,\mathrm{d}t\right]^2 \tag{58}$$

$$= \frac{1}{s^2}\left[\int_0^1 f(px)\,\mathrm{d}p - \int_1^\infty f(px)\,\mathrm{d}p + 2\int_1^s f(px)\,\mathrm{d}p\right]^2 \tag{59}$$

$$= \frac{1}{s^2}\left[g(x)^2 + 4g(x)\int_1^s f(px)\,\mathrm{d}p + 4\left(\int_1^s f(px)\,\mathrm{d}p\right)^2\right]. \tag{60}$$

The last equation leads to the condition:

$$[g(x) + p_x(s)] \cdot p_x(s) \propto (s^2 - 1) \tag{61}$$

where

$$p_x(s) = \int_1^s f(tx)\,\mathrm{d}t. \tag{62}$$

In order to satisfy equation (61) function $p_x(s)$ has to behave asymptotically as s reaches the infinity limit. Consequently the second integral in (57) becomes infinite as it is equal to $p_x(\infty)$. The argumentation above leads to the conclusion that there are only two classes of functions which can satisfy equation (54). One is the constant null vector for α and β and the other is a singular configuration concentrated at the origin of the coordinate system. By rewriting formula for $g(sx)$ for such a singular configuration we obtain for $s > 0$:

$$g(sx) = \frac{1}{s}\left[\int_0^1 f(px)\,\mathrm{d}p + \int_1^s f(px)\,\mathrm{d}p\right]. \tag{63}$$

As the term with integrals of f from 1 to s vanishes, this leads to the scaling law for $g(x)$:

$$g(sx) = (1/s)g(x) \tag{64}$$

and an obvious solution for $g(x)$:

$$g(x) = v_0/|x| \neq \text{constant} \tag{65}$$

which contradicts (54) and thus cannot be a solution of the system of equations (51). This leads us to the final conclusion that there are no solutions of the system of equations (51) with non-trivial Hopf index.

5. Conclusions

We have proved that there are no configurations with minimal energy in a given, non-trivial Hopf class for a Heisenberg Hamiltonian with Hopf index as a topological term. Only time independent configurations were considered (There still may be time-dependent configurations with non-trivial Hopf index.) This result may be interpreted in the following way. An energy functional may have a minimum for an 'infinitesimally flat' configuration, i.e. the one with infinitesimal difference from the constant vector but still possessing a non-zero Hopf index. We have also shown that the use of derivative coordinates (like the one in (17)) could be effective in the search for conditional extremes of the Hamiltonian with a topological term. This method could be used as an alternative to Bogomolny separation as it is usually easier to find an appropriate set of variables than perform the separation. The method can also be applied to other cases. Obvious applications include the nonlinear σ-models. For instance in [10] authors use the O(3) σ-model as a basis for extended action which is shown to be equivalent to the spin-s theory. Since a Euclidean metric is used, the base action appear to be identical to (48). Our result shows that this base action has no stationary points.

Some connection with the Skyrme model could also be considered, particularly with a known Hopf soliton [11]. In fact these two theories appear to have little in common. Skyrme theory is $S^3 \mapsto S^3$, not $S^3 \mapsto S^2$. Furthermore, the Hopf soliton is an $S^3 \mapsto S^3$ *map derived from an* $S^3 \mapsto S^2$ *one, and thus it can evolve into a configuration without a* component with a non-zero Hopf index.

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